

# STRONG TREE PROPERTIES FOR TWO SUCCESSIVE CARDINALS

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**ABSTRACT.** An inaccessible cardinal  $\kappa$  is supercompact when  $(\kappa, \lambda)$ -ITP holds for all  $\lambda \geq \kappa$ . We prove that if there is a model of ZFC with two supercompact cardinals, then there is a model of ZFC where simultaneously  $(\aleph_2, \mu)$ -ITP and  $(\aleph_3, \mu')$ -ITP hold, for all  $\mu \geq \aleph_2$  and  $\mu' \geq \aleph_3$ .

**Key Words:** tree property, large cardinals, forcing.

**Mathematical Subject Classification:** 03E55.

## 1. INTRODUCTION

The result presented in this paper concern two combinatorial properties that generalize the usual tree property for a regular cardinal. It is a well known fact that an inaccessible cardinal is weakly compact if, and only if, it satisfies the tree property. A similar characterization was made by Jech [3] and Magidor [7] for strongly compact and supercompact cardinals; we will refer to the corresponding combinatorial properties as the *strong tree property* and the *super tree property*. Thus, an inaccessible cardinal is strongly compact if, and only if, it satisfies the strong tree property (see Jech [3]), while it is supercompact if, and only if, it satisfies the super tree property (see Magidor [7]).

While the previous results date to the early 1970s, it was only recently that a systematic study of these properties was undertaken by Weiss (see [11] and [12]). Although the strong tree property and the super tree property characterize large cardinals, they can be satisfied by small cardinals as well. Indeed, Weiss proved in [12] that for every  $n \geq 2$ , one can define a model of the super tree property for  $\aleph_n$ , starting from a model with a supercompact cardinal. Since the super tree property captures the combinatorial essence of supercompact cardinals, then we can say that in Weiss model,  $\aleph_n$  is in some sense supercompact.

By working on the super tree property at  $\aleph_2$ , Viale and Weiss (see [10] and [9]) obtained new results about the consistency strength of the Proper Forcing Axiom. They proved that if one forces a model of PFA using a forcing that collapse  $\kappa$  to  $\omega_2$  and

satisfies the  $\kappa$ -covering and the  $\kappa$ -approximation properties, then  $\kappa$  has to be strongly compact; if the forcing is also proper, then  $\kappa$  is supercompact. Since every known forcing producing a model of PFA by collapsing  $\kappa$  to  $\omega_2$  satisfies those conditions, we can say that the consistency strength of PFA is, reasonably, a supercompact cardinal.

It is natural to ask whether two small cardinals can simultaneously have the strong or the super tree properties. Abraham define in [1] a forcing construction producing a model of the tree property for  $\aleph_2$  and  $\aleph_3$ , starting from a model of ZFC + GCH with a supercompact cardinal and a weakly compact cardinal above it. Cummings and Foreman [2] proved that if there is a model of set theory with infinitely many supercompact cardinals, then one can obtain a model in which every  $\aleph_n$  with  $n \geq 2$  satisfies the tree property.

In the present paper, we construct a model of set theory in which  $\aleph_2$  and  $\aleph_3$  simultaneously satisfy the super tree property, starting from a model of ZFC with two supercompact cardinals  $\kappa < \lambda$ . We will collapse  $\kappa$  to  $\aleph_2$  and  $\lambda$  to  $\aleph_3$ , in such a way that they will still satisfy the super tree property. The definition of the forcing construction required for that theorem is motivated by Abraham [1] and Cummings-Foreman [2]. We also conjecture that in the model defined by Cummings and Foreman, every  $\aleph_n$  (with  $n \geq 2$ ) satisfies the super tree property.

The paper is organized as follows. In §3 we introduce the strong and the super tree properties. In §5, §6 and §7 we define the forcing notion required for the final theorem and we discuss some properties of that forcing. §4 is devoted to the proof of two preservation theorems. Finally, the proof of the main theorem is developed in §8.

## 2. PRELIMINARIES AND NOTATION

Given a forcing  $\mathbb{P}$  and conditions  $p, q \in \mathbb{P}$ , we use  $p \leq q$  in the sense that  $p$  is stronger than  $q$ . A poset  $\mathbb{P}$  is *separative* if whenever  $q \not\leq p$ , then some extension of  $q$  in  $\mathbb{P}$  is incompatible with  $p$ . Every partial order can be turned into a separative poset. Indeed, one can define  $p \prec q$  iff all extensions of  $p$  are compatible with  $q$ , then the resulting equivalence relation, given by  $p \sim q$  iff  $p \prec q$  and  $q \prec p$ , provides a separative poset; we denote by  $[p]$  the equivalence class of  $p$ .

Given two forcings  $\mathbb{P}$  and  $\mathbb{Q}$ , we will write  $\mathbb{P} \equiv \mathbb{Q}$  when  $\mathbb{P}$  and  $\mathbb{Q}$  are equivalent, namely:

- (1) for every filter  $G_{\mathbb{P}} \subseteq \mathbb{P}$  which is  $V$ -generic over  $\mathbb{P}$ , there exists a filter  $G_{\mathbb{Q}} \subseteq \mathbb{Q}$  which is  $V$ -generic over  $\mathbb{Q}$  and  $V[G_{\mathbb{P}}] = V[G_{\mathbb{Q}}]$ ;
- (2) for every filter  $G_{\mathbb{Q}} \subseteq \mathbb{Q}$  which is  $V$ -generic over  $\mathbb{Q}$ , there exists a filter  $G_{\mathbb{P}} \subseteq \mathbb{P}$  which is  $V$ -generic over  $\mathbb{P}$  and  $V[G_{\mathbb{P}}] = V[G_{\mathbb{Q}}]$ .

If  $\mathbb{P}$  is any forcing and  $\dot{\mathbb{Q}}$  is a  $\mathbb{P}$ -name for a forcing, then we denote by  $\mathbb{P} * \dot{\mathbb{Q}}$  the poset  $\{(p, q); p \in \mathbb{P}, q \in V^{\mathbb{P}} \text{ and } p \Vdash q \in \dot{\mathbb{Q}}\}$ , where for every  $(p, q), (p', q') \in \mathbb{P} * \dot{\mathbb{Q}}$ ,  $(p, q) \leq (p', q')$  if, and only if,  $p \leq p'$  and  $p \Vdash q \leq q'$ .

If  $\mathbb{P}$  and  $\mathbb{Q}$  are two posets, a *projection*  $\pi : \mathbb{Q} \rightarrow \mathbb{P}$  is a function such that:

- (1) for all  $q, q' \in \mathbb{Q}$ , if  $q \leq q'$ , then  $\pi(q) \leq \pi(q')$ ;
- (2)  $\pi(1_{\mathbb{Q}}) = 1_{\mathbb{P}}$ ;
- (3) for all  $q \in \mathbb{Q}$ , if  $p \leq \pi(q)$ , then there is  $q' \leq q$  such that  $\pi(q') \leq p$ .

If  $\pi : \mathbb{Q} \rightarrow \mathbb{P}$  is a projection and  $G_{\mathbb{P}} \subseteq \mathbb{P}$  is a  $V$ -generic filter, define

$$\mathbb{Q}/G_{\mathbb{P}} := \{q \in \mathbb{Q}; \pi(q) \in G_{\mathbb{P}}\},$$

$\mathbb{Q}/G_{\mathbb{P}}$  is ordered as a subposet of  $\mathbb{Q}$ . The following hold:

- (1) If  $G_{\mathbb{Q}} \subseteq \mathbb{Q}$  is a generic filter over  $V$  and  $H := \{p \in \mathbb{P}; \exists q \in G_{\mathbb{Q}}(\pi(q) \leq p)\}$ , then  $H$  is  $\mathbb{P}$ -generic over  $V$ ;
- (2) if  $G_{\mathbb{P}} \subseteq \mathbb{P}$  is a generic filter over  $V$ , and if  $G \subseteq \mathbb{Q}/G_{\mathbb{P}}$  is a generic filter over  $V[G_{\mathbb{P}}]$ , then  $G$  is  $\mathbb{Q}$ -generic over  $V$ , and  $\pi''[G]$  generates  $G_{\mathbb{P}}$ ;
- (3) if  $G_{\mathbb{Q}} \subseteq \mathbb{Q}$  is a generic filter, and  $H := \{p \in \mathbb{P}; \exists q \in G_{\mathbb{Q}}(\pi(q) \leq p)\}$ , then  $G_{\mathbb{Q}}$  is  $\mathbb{Q}/G_{\mathbb{P}}$ -generic over  $V[H]$ . That is, we can factor forcing with  $\mathbb{Q}$  as forcing with  $\mathbb{P}$  followed by forcing with  $\mathbb{Q}/G_{\mathbb{P}}$  over  $V[G_{\mathbb{P}}]$ .

Some of our projections  $\pi : \mathbb{Q} \rightarrow \mathbb{P}$  will also have the following property: for all  $p \leq \pi(q)$ , there is  $q' \leq q$  such that

- (1)  $\pi(q') = p$ ,
- (2) for every  $q^* \leq q$ , if  $\pi(q^*) \leq p$ , then  $q^* \leq q'$ .

We denote by  $\text{ext}(q, p)$  any condition like  $q'$  above (if a condition  $q''$  satisfies the previous properties, then  $q' \leq q'' \leq q$ ). In this case, if  $G_{\mathbb{P}} \subseteq \mathbb{P}$  is a generic filter, we can define an ordering on  $\mathbb{Q}/G_{\mathbb{P}}$  as follows:  $p \leq^* q$  if, and only if, there is  $r \leq \pi(p)$  such that  $r \in G_{\mathbb{P}}$  and  $\text{ext}(p, r) \leq q$ . Then, forcing over  $V[G_{\mathbb{P}}]$  with  $\mathbb{Q}/G_{\mathbb{P}}$  ordered as a subposet of  $\mathbb{Q}$ , is equivalent to forcing over  $V[G_{\mathbb{P}}]$  with  $(\mathbb{Q}/G_{\mathbb{P}}, \leq^*)$ .

Let  $\kappa$  be a regular cardinal and  $\lambda$  an ordinal, we denote by  $\text{Add}(\kappa, \lambda)$  the poset of all partial functions  $f : \lambda \rightarrow 2$  of size less than  $\kappa$ , ordered by reverse inclusion. We use  $\text{Add}(\kappa)$  to denote  $\text{Add}(\kappa, \kappa)$ .

If  $V \subseteq W$  are two models of set theory with the same ordinals and  $\eta$  is a cardinal in  $W$ , we say that  $(V, W)$  has the  $\eta$ -covering property if, and only if, every set  $X \subseteq V$  in  $W$  of cardinality less than  $\eta$  in  $W$ , is contained in a set  $Y \in V$  of cardinality less than  $\eta$  in  $V$ .

Assume that  $\mathbb{P}$  is a forcing notion, we will use  $\langle \mathbb{P} \rangle$  to denote the canonical  $\mathbb{P}$ -name for a  $\mathbb{P}$ -generic filter over  $V$ .

**Lemma 2.1.** (*Easton's Lemma*) *Let  $\kappa$  be regular. If  $\mathbb{P}$  has the  $\kappa$ -chain condition and  $\mathbb{Q}$  is  $\kappa$ -closed, then*

- (1)  $\Vdash_{\mathbb{Q}} \mathbb{P}$  *has the  $\kappa$ -chain condition;*
- (2)  $\Vdash_{\mathbb{P}} \mathbb{Q}$  *is a  $< \kappa$ -distributive;*
- (3) *If  $G$  is  $\mathbb{P}$ -generic over  $V$  and  $H$  is  $\mathbb{Q}$ -generic over  $V$ , then  $G$  and  $H$  are mutually generic;*
- (4) *If  $G$  is  $\mathbb{P}$ -generic over  $V$  and  $H$  is  $\mathbb{Q}$ -generic over  $V$ , then  $(V, V[G][H])$  has the  $\kappa$ -covering property;*
- (5) *If  $\mathbb{R}$  is  $\kappa$ -closed, then  $\Vdash_{\mathbb{P} \times \mathbb{Q}} \mathbb{R}$  is  $< \kappa$ -distributive.*

For a proof of that lemma see [2, Lemma 2.11].

Let  $\eta$  be a regular cardinal,  $\theta > \eta$  be large enough and  $M \prec H_\theta$  of size  $\eta$ . We say that  $M$  is *internally approachable of length  $\eta$*  if it can be written as the union of an increasing continuous chain  $\langle M_\xi : \xi < \eta \rangle$  of elementary submodels of  $H(\theta)$  of size less than  $\eta$ , such that  $\langle M_\xi : \xi < \eta' \rangle \in M_{\eta'+1}$ , for every ordinal  $\eta' < \eta$ .

**Lemma 2.2.** ( $\Delta$ -system Lemma) *Assume that  $\lambda$  is a regular cardinal and  $\kappa < \lambda$  is such that  $\alpha^{<\kappa} < \lambda$ , for every  $\alpha < \lambda$ . Let  $\mathcal{F}$  be a family of sets of cardinality less than  $\kappa$  such that  $|\mathcal{F}| = \lambda$ . There exists a family  $\mathcal{F}' \subseteq \mathcal{F}$  of size  $\lambda$  and a set  $R$  such that  $X \cap Y = R$ , for any two distinct  $X, Y \in \mathcal{F}'$ .*

For a proof of that lemma see [5].

**Lemma 2.3.** (*Pressing Down Lemma*) *If  $f$  is a regressive function on a stationary set  $S \subseteq [A]^{<\kappa}$  (i.e.  $f(x) \in x$ , for every non empty  $x \in S$ ), then there exists a stationary set  $T \subseteq S$  such that  $f$  is constant on  $T$ .*

For a proof of that lemma see [5].

We will assume familiarity with the theory of large cardinals and elementary embeddings, as developed for example in [4].

**Lemma 2.4.** (*Laver*) [6] *If  $\kappa$  is a supercompact cardinal, then there exists  $L : \kappa \rightarrow V_\kappa$  such that: for all  $\lambda$ , for all  $x \in H_{\lambda^+}$ , there is  $j : V \rightarrow M$  such that  $j(\kappa) > \lambda$ ,  ${}^\lambda M \subseteq M$  and  $j(L)(\kappa) = x$ .*

**Lemma 2.5.** (*Silver*) *Let  $j : M \rightarrow N$  be an elementary embedding between inner models of ZFC. Let  $\mathbb{P} \in M$  be a forcing and suppose that  $G$  is  $\mathbb{P}$ -generic over  $M$ ,  $H$  is  $j(\mathbb{P})$ -generic over  $N$ , and  $j''[G] \subseteq H$ . Then, there is a unique  $j^* : M[G] \rightarrow N[H]$  such that  $j^* \upharpoonright M = j$  and  $j^*(G) = H$ .*

*Proof.* If  $j''[G] \subseteq H$ , then the map  $j^*(\dot{x}^G) = j(\dot{x})^H$  is well defined and satisfies the required properties.  $\square$

## 3. THE STRONG AND THE SUPER TREE PROPERTIES

In order to define the strong tree property and the super tree property for a regular cardinal  $\kappa \geq \aleph_2$ , we need to define the notion of  $(\kappa, \lambda)$ -tree, for an ordinal  $\lambda \geq \kappa$ .

**Definition 3.1.** *Given  $\kappa \geq \omega_2$  a regular cardinal and  $\lambda \geq \kappa$ , a  $(\kappa, \lambda)$ -tree is a set  $F$  satisfying the following properties:*

- (1) *for every  $f \in F$ ,  $f : X \rightarrow 2$ , for some  $X \in [\lambda]^{<\kappa}$*
- (2) *for all  $f \in F$ , if  $X \subseteq \text{dom}(f)$ , then  $f \restriction X \in F$ ;*
- (3) *the set  $\text{Lev}_X(F) := \{f \in F; \text{dom}(f) = X\}$  is non empty, for all  $X \in [\lambda]^{<\kappa}$ ;*
- (4)  *$|\text{Lev}_X(F)| < \kappa$ , for all  $X \in [\lambda]^{<\kappa}$ .*

When there is no ambiguity, we will simply write  $\text{Lev}_X$  instead of  $\text{Lev}_X(F)$ .

**Definition 3.2.** *Given  $\kappa \geq \omega_2$  a regular cardinal,  $\lambda \geq \kappa$ , and  $F$  a  $(\kappa, \lambda)$ -tree,*

- (1) *a cofinal branch for  $F$  is a function  $b : \lambda \rightarrow 2$  such that  $b \restriction X \in \text{Lev}_X(F)$ , for all  $X \in [\lambda]^{<\kappa}$ ;*
- (2) *an  $F$ -level sequence is a function  $D : [\lambda]^{<\kappa} \rightarrow F$  such that for every  $X \in [\lambda]^{<\kappa}$ ,  $D(X) \in \text{Lev}_X(F)$ ;*
- (3) *given an  $F$ -level sequence  $D$ , an ineffable branch for  $D$  is a cofinal branch  $b : \lambda \rightarrow 2$  such that  $\{X \in [\lambda]^{<\kappa}; b \restriction X = D(X)\}$  is stationary.*

**Definition 3.3.** *Given  $\kappa \geq \omega_2$  a regular cardinal and  $\lambda \geq \kappa$ ,*

- (1)  *$(\kappa, \lambda)$ -TP holds if every  $(\kappa, \lambda)$ -tree has a cofinal branch;*
- (2)  *$(\kappa, \lambda)$ -ITP holds if for every  $(\kappa, \lambda)$ -tree  $F$  and for every  $F$ -level sequence  $D$ , there is an ineffable branch for  $D$ ;*
- (3) *we say that  $\kappa$  satisfies the strong tree property if  $(\kappa, \mu)$ -TP holds, for all  $\mu \geq \kappa$ ;*
- (4) *we say that  $\kappa$  satisfies the super tree property if  $(\kappa, \mu)$ -ITP holds, for all  $\mu \geq \kappa$ ;*

## 4. THE PRESERVATION THEOREMS

It will be important in what follows that certain forcings cannot add ineffable branches.

**Theorem 4.1.** *(First Preservation Theorem) Let  $\theta$  be a regular cardinal and  $\mu \geq \theta$  be any ordinal. Assume that  $F$  is a  $(\theta, \mu)$ -tree and  $\mathbb{Q}$  is an  $\eta^+$ -closed forcing with  $\eta < \theta \leq 2^\eta$ . For every filter  $G_{\mathbb{Q}} \subseteq \mathbb{Q}$  generic over  $V$ , every cofinal branch for  $F$  in  $V[G_{\mathbb{Q}}]$  is already in  $V$ .*

*Proof.* We can assume, without loss of generality, that  $\eta$  is minimal such that  $2^\eta \geq \theta$ . Assume towards a contradiction that  $\mathbb{Q}$  adds a cofinal branch to  $F$ , let  $\dot{b}$  be a  $\mathbb{Q}$ -name for such a function. For all  $\alpha \leq \eta$  and all  $s \in {}^\alpha 2$ , we are going to define by induction three objects  $a_\alpha \in [\mu]^{<\theta}$ ,  $f_s \in \text{Lev}_{a_\alpha}$  and  $p_s \in \mathbb{Q}$  such that:

- (1)  $p_s \Vdash \dot{b} \restriction a_\alpha = f_s$ ;
- (2)  $f_{s \smallfrown 0}(\beta) \neq f_{s \smallfrown 1}(\beta)$ , for some  $\beta < \mu$ ;

- (3) if  $s \subseteq t$ , then  $p_t \leq p_s$ ;
- (4) if  $\alpha < \beta$ , then  $a_\alpha \subset a_\beta$ .

Let  $\alpha < \eta$ , assume that  $a_\alpha, f_s$  and  $p_s$  have been defined for all  $s \in {}^\alpha 2$ . We define  $a_{\alpha+1}, f_s$ , and  $p_s$ , for all  $s \in {}^{\alpha+1} 2$ . Let  $t$  be in  ${}^\alpha 2$ , we can find an ordinal  $\beta_t \in \mu$  and two conditions  $p_{t \smallfrown 0}, p_{t \smallfrown 1} \leq p_t$  such that  $p_{t \smallfrown 0} \Vdash \dot{b}(\beta_t) = 0$  and  $p_{t \smallfrown 1} \Vdash \dot{b}(\beta_t) = 1$ . (otherwise,  $\dot{b}$  would be a name for a cofinal branch which is already in  $V$ ). Let  $a_{\alpha+1} := a_\alpha \cup \{\beta_t; t \in {}^\alpha 2\}$ , then  $|a_{\alpha+1}| < \theta$ , because  $2^\alpha < \theta$ . We just defined, for every  $s \in {}^{\alpha+1} 2$ , a condition  $p_s$ . Now, by strengthening  $p_s$  if necessary, we can find  $f_s \in \text{Lev}_{a_{\alpha+1}}$  such that

$$p_s \Vdash \dot{b} \restriction a_{\alpha+1} = f_s.$$

Finally,  $f_{t \smallfrown 0}(\beta_t) \neq f_{t \smallfrown 1}(\beta_t)$ , for all  $t \in {}^\alpha 2$ : because  $p_{t \smallfrown 0} \Vdash f_{t \smallfrown 0}(\beta_t) = \dot{b}(\beta_t) = 0$ , while  $p_{t \smallfrown 1} \Vdash f_{t \smallfrown 1}(\beta_t) = \dot{b}(\beta_t) = 1$ .

If  $\alpha$  is a limit ordinal  $\leq \eta$ , let  $t$  be any function in  ${}^\alpha 2$ . Since  $\mathbb{Q}$  is  $\eta^+$ -closed, there is a condition  $p_t$  such that  $p_t \leq p_{t \restriction \beta}$ , for all  $\beta < \alpha$ . Define  $a_\alpha := \bigcup_{\beta < \alpha} a_\beta$ . By strengthening  $p_t$  if necessary, we can find  $f_t \in \text{Lev}_{a_\alpha}$  such that  $p_t \Vdash \dot{b} \restriction a_\alpha = f_t$ . That completes the construction.

We show that  $|\text{Lev}_{a_\eta}| \geq {}^\eta 2 \geq \theta$ , thus a contradiction is obtained. Let  $s \neq t$  be two functions in  ${}^\eta 2$ , we are going to prove that  $f_s \neq f_t$ . Let  $\alpha$  be the minimum ordinal less than  $\eta$  such that  $s(\alpha) \neq t(\alpha)$ , without loss of generality  $r \smallfrown 0 \sqsubset s$  and  $r \smallfrown 1 \sqsubset t$ , for some  $r \in {}^\alpha 2$ . By construction,

$$p_s \leq p_{r \smallfrown 0} \Vdash \dot{b} \restriction a_{\alpha+1} = f_{r \smallfrown 0} \text{ and } p_t \leq p_{r \smallfrown 1} \Vdash \dot{b} \restriction a_{\alpha+1} = f_{r \smallfrown 1},$$

where  $f_{r \smallfrown 0}(\beta) \neq f_{r \smallfrown 1}(\beta)$ , for some  $\beta$ . Moreover,  $p_s \Vdash \dot{b} \restriction a_\eta = f_s$  and  $p_t \Vdash \dot{b} \restriction a_\eta = f_t$ , hence  $f_s \restriction a_{\alpha+1}(\beta) = f_{r \smallfrown 0}(\beta) \neq f_{r \smallfrown 1}(\beta) = f_t \restriction a_{\alpha+1}(\beta)$ , thus  $f_s \neq f_t$ . That completes the proof.  $\square$

**Corollary 4.2.** *Let  $\theta$  be a regular cardinal and  $\mu \geq \theta$  be any ordinal. Assume that  $F$  is a  $(\theta, \mu)$ -tree and  $D$  is an  $F$ -level sequence, and suppose that  $\mathbb{Q}$  is an  $\eta^+$ -closed forcing with  $\eta < \theta \leq 2^\eta$ . For every filter  $G_\mathbb{Q} \subseteq \mathbb{Q}$  generic over  $V$ , if  $D$  has no ineffable branches in  $V$ , then there are no ineffable branches for  $D$  in  $V[G_\mathbb{Q}]$ .*

*Proof.* Assume that  $b \in V[G_\mathbb{Q}]$  is an ineffable branch for  $D$ . By Proposition 4.1, we have  $b \in V$ . Define in  $V$  the set

$$S := \{X \in [\mu]^{<\theta}; b \restriction X = D(X)\}.$$

Then,  $S$  is stationary in  $V[G_\mathbb{Q}]$ , hence it is stationary in  $V$ . Thus  $b$  is an ineffable branch for  $D$  in  $V$ .  $\square$

The following proposition is rather ad hoc. It will be used several times in the final theorem.

**Theorem 4.3.** (*Second Preservation Theorem*) *Let  $V \subseteq W$  be two models of set theory with the same ordinals and let  $\mathbb{P} \in V$  be a forcing notion and  $\kappa$  a cardinal in  $V$  such that:*

- (1)  $\mathbb{P} \subseteq \text{Add}(\aleph_n, \tau)^V$ , for some  $\tau > \aleph_n$ ,  
and for every  $p \in \mathbb{P}$ , if  $X \subseteq \text{dom}(p)$ , then  $p \restriction X \in \mathbb{P}$ ,
- (2)  $\aleph_n^V = \aleph_n^W$ , for every  $m \leq n$ , and  $W \models \kappa = \aleph_{n+1}$ ,
- (3)  $(V, W)$  has the  $\kappa$ -covering property,
- (4) in  $V$ , we have  $\gamma^{<\aleph_n} < \kappa$ , for every cardinal  $\gamma < \kappa$ .

*Let  $F \in W$  be a  $(\aleph_{n+1}, \mu)$ -tree with  $\mu \geq \aleph_{n+1}$ , then for every filter  $G_{\mathbb{P}} \subseteq \mathbb{P}$  generic over  $W$ , every cofinal branch for  $F$  in  $W[G_{\mathbb{P}}]$  is already in  $W$ .*

*Proof.* Work in  $W$ . Let  $\dot{b} \in W^{\mathbb{P}}$  and let  $p \in \mathbb{P}$  such that

$$p \Vdash \dot{b} \text{ is a cofinal branch for } F.$$

We are going to find a condition  $q \in \mathbb{P}$  such that  $q \parallel p$  and for some  $b \in W$ , we have  $q \Vdash \dot{b} = b$ . Let  $\chi$  be large enough, for all  $X \prec H_\chi$  of size  $\aleph_n$ , we fix a condition  $p_X \leq p$  and a function  $f_X \in \text{Lev}_{X \cap \mu}$  such that

$$p_X \Vdash \dot{b} \restriction X = f_X.$$

Let  $S$  be the set of all the structures  $X \prec H_\chi$ , such that  $X$  is internally approachable of length  $\aleph_n$ . Since every condition of  $\mathbb{P}$  has size less than  $\aleph_n$ , then for all  $X \in S$ , there is  $M_X \in X$  of size less than  $\aleph_n$  such that

$$p_X \restriction X \subseteq M_X.$$

By the Pressing Down Lemma, there exists  $M^*$  and a stationary set  $E^* \subseteq S$  such that  $M^* = M_X$ , for all  $X \in E^*$ . The set  $M^*$  has size less than  $\kappa$  in  $W$ , hence  $A := \bigcup_{X \in E^*} p_X \restriction M^*$  has size  $< \kappa$  in  $W$ . By the assumption,  $A$  is covered by some  $N \in V$  of size  $\gamma < \kappa$  in  $V$ . In  $V$ , we have  $|[N]^{<\aleph_n}| = \gamma^{<\aleph_n} < \kappa$ . It follows that in  $W$  there are less than  $\kappa$  (hence less than  $\aleph_{n+1}$ ) possible values for  $p_X \restriction M^*$ . Therefore, we can find in  $W$  a cofinal  $E \subseteq E^*$  and a condition  $q \in \mathbb{P}$ , such that  $p_X \restriction X = q$ , for all  $X \in E$ .

**Claim 4.4.**  $f_X \restriction Y = f_Y \restriction X$ , for all  $X, Y \in E$ .

*Proof.* Let  $X, Y \in E$ , there is  $Z \in E$  with  $X, Y, \text{dom}(p_X), \text{dom}(p_Y) \subseteq Z$ . Then, we have  $p_X \cap p_Z = p_X \cap p_Z \restriction Z = p_X \cap q = q$ , thus  $p_X \parallel p_Z$  and similarly  $p_Y \parallel p_Z$ . Let  $r \leq p_X, p_Z$  and  $s \leq p_Y, p_Z$ , then  $r \Vdash f_Z \restriction X = \dot{b} \restriction X = f_X$  and  $s \Vdash f_Z \restriction Y = \dot{b} \restriction Y = f_Y$ . It follows that  $f_X \restriction Y = f_Z \restriction (X \cap Y) = f_Y \restriction X$ .  $\square$

Let  $b$  be  $\bigcup_{X \in E} f_X$ . The previous claim implies that  $b$  is a function and

$$b \restriction X = f_X, \text{ for all } X \in E.$$

**Claim 4.5.**  $q \Vdash \dot{b} = b$ .

*Proof.* We show that for every  $X \in E$ , the set  $B_X := \{s \in \mathbb{P}; s \Vdash \dot{b} \restriction X = b \restriction X\}$  is dense below  $q$ . Let  $r \leq q$ , there is  $Y \in E$  such that  $\text{dom}(r), X \subseteq Y$ . It follows that  $p_Y \cap r = p_Y \restriction Y \cap r = q \cap r = q$ , thus  $p_Y \Vdash r$ . Let  $s \leq p_Y, r$ , then  $s \in B_X$ , because  $s \Vdash \dot{b} \restriction X = f_Y \restriction X = f_X = b \restriction X$ . Since  $\bigcup\{X \cap \mu; X \in E\} = \mu$ , we have  $q \Vdash \dot{b} = b$ .  $\square$

That completes the proof.  $\square$

**Theorem 4.6.** *Let  $V \subseteq W$  be two models of set theory with the same ordinals and let  $\mathbb{P} \in V$  be a forcing notion and  $\kappa$  a cardinal in  $V$  such that:*

- (1)  $\mathbb{P} \subseteq \text{Add}(\aleph_n, \tau)^V$ , for some  $\tau > \aleph_n$ ,  
and for every  $p \in \mathbb{P}$ , if  $X \subseteq \text{dom}(p)$ , then  $p \restriction X \in \mathbb{P}$ ,
- (2)  $\aleph_m^V = \aleph_m^W$ , for every  $m \leq n$ , and  $W \models |\kappa| = \aleph_{n+1}$ ,
- (3) for every set  $X \subseteq V$  in  $W$  of size  $< \aleph_{n+1}$  in  $W$ , there is  $Y \in V$  of size  $< \kappa$  in  $V$ , such that  $X \subseteq Y$ ,
- (4) in  $V$ , we have  $\gamma^{<\aleph_n} < \kappa$ , for every cardinal  $\gamma < \kappa$ .

Let  $F \in W$  be a  $(\aleph_{n+1}, \mu)$ -tree with  $\mu \geq \aleph_{n+1}$ , then for every filter  $G_{\mathbb{P}} \subseteq \mathbb{P}$  generic over  $W$ , every cofinal branch for  $F$  in  $W[G_{\mathbb{P}}]$  is already in  $W$ .

*Proof.* Same proof as for the Second Preservation Theorem.  $\square$

## 5. THE MAIN FORCING

**Definition 5.1.** *Let  $\eta$  be a regular cardinal and let  $\theta$  be any ordinal, we define*

$$\mathbb{P}(\eta, \theta) := \{p \in \text{Add}(\eta, \theta); \text{ for every } \alpha \in \text{dom}(p), \alpha \text{ is a successor ordinal} \},$$

$\mathbb{P}(\eta, \theta)$  is ordered by reverse inclusion.

For  $E \subseteq \theta$ , we denote by  $P(\eta, \theta) \restriction E$  the set of all functions in  $P(\eta, \theta)$  with domain a subset of  $E$ . The following definition is due to Abraham [1].

**Definition 5.2.** *Assume that  $V \subseteq W$  are two models of set theory with the same ordinals, let  $\eta$  be a regular cardinal in  $W$  and let  $\mathbb{P} := \mathbb{P}(\eta, \theta)^V$ , where  $\theta$  is any ordinal. We define in  $W$  the poset  $\mathbb{M}(\eta, \theta, V, W)$  as follows:*

$(p, q) \in \mathbb{M}(\eta, \theta, V, W)$  if, and only if,

- (1)  $p \in \mathbb{P}(\eta, \theta)^V$ ;
- (2)  $q \in W$  is a partial function on  $\theta$  of size  $\leq \eta$  such that for every  $\alpha \in \text{dom}(q)$ ,  $\alpha$  is a successor ordinal,  $q(\alpha) \in W^{\mathbb{P} \restriction \alpha}$ , and  $\Vdash_{\mathbb{P} \restriction \alpha}^W q(\alpha) \in \text{Add}(\eta^+, \theta)^{V[\langle \mathbb{P} \restriction \alpha \rangle]}$ .

$\mathbb{M}(\eta, \theta, V, W)$  is partially ordered by  $(p, q) \leq (p', q')$  if, and only if,

- (1)  $p \leq p'$ ;
- (2)  $\text{dom}(q') \subseteq \text{dom}(q)$ ;
- (3)  $p \restriction \alpha \Vdash_{\mathbb{P} \restriction \alpha}^W q(\alpha) \leq q'(\alpha)$ , for all  $\alpha \in \text{dom}(q')$ .



If  $\theta$  is a weakly compact cardinal, then  $\mathbb{M}(\aleph_n, \theta, V, V)$  corresponds to the forcing defined by Mitchell for a model of the tree property at  $\aleph_{n+2}$  (see [8]). Weiss proved that a variation of that forcing with  $\theta$  supercompact, produces a model of the super tree property for  $\aleph_{n+2}$ . Let's discuss a naive attempt to build a model of the super tree property for two successive cardinals  $\aleph_n, \aleph_{n+1}$  (with  $n \geq 2$ ). We start with two supercompact cardinals  $\kappa < \lambda$  in a model  $V$ , then we force with  $\mathbb{M}(\aleph_{n-2}, \kappa, V, V)$  over  $V$  obtaining a model  $W$ ; finally, we force over  $W$  with  $\mathbb{M}(\aleph_{n-1}, \lambda, W, W)$ . The problem with this approach is that the second stage might introduce an  $(\aleph_n, \mu)$ -tree  $F$  with no cofinal branches. Therefore, we have to define the first stage of the iteration so that it will make the super tree property at  $\aleph_n$  “indestructible”. The forcing notion required for that “anticipates” at the first stage a fragment of  $\mathbb{M}(\aleph_{n-1}, \lambda, W, W)$ .

**Definition 5.3.** For  $V, W$  and  $\eta, \theta$  like in Definition 5.2, we define

$$\mathbb{Q}^*(\eta, \theta, V, W) := \{(\emptyset, q); (\emptyset, q) \in \mathbb{M}(\eta, \theta, V, W)\}.$$

The poset defined hereafter is a variation of the forcing construction defined by Abraham in [1, Definition 2.14].

**Definition 5.4.** Let  $V$  be a model of set theory, and suppose that  $\theta > \aleph_n$  is an inaccessible cardinal. Let  $\mathbb{P} := \mathbb{P}(\aleph_n, \theta)^V$  and let  $L : \theta \rightarrow V_\theta$  be any function. Define

$$\mathbb{R} := \mathbb{R}(\aleph_n, \theta, L)$$

as follows. For each  $\beta \leq \theta$ , we define by induction  $\mathbb{R} \restriction \beta$  and then we set  $\mathbb{R} = \mathbb{R} \restriction \kappa$ .

$\mathbb{R} \restriction 0$  is the trivial forcing.

$(p, q, f) \in \mathbb{R} \restriction \beta$  if, and only if

- (1)  $p \in \mathbb{P} \restriction \beta (= \mathbb{P}(\aleph_n, \beta)^V)$ ;
- (2)  $q$  is a partial function on  $\beta$  of size  $\leq \aleph_n$ , such that for every  $\alpha \in \text{dom}(q)$ ,  $\alpha$  is a successor ordinal,  $q(\alpha) \in V^{\mathbb{P} \restriction \alpha}$  and  $\Vdash_{\mathbb{P} \restriction \alpha} q(\alpha) \in \text{Add}(\aleph_{n+1})^{V[\mathbb{P} \restriction \alpha]}$ ;
- (3)  $f$  is a partial function on  $\beta$  of size  $\leq \aleph_n$  such that for all  $\alpha \in \text{dom}(f)$ ,  $\alpha$  is a limit ordinal,  $f(\alpha) \in V^{\mathbb{R} \restriction \alpha}$  and

$\Vdash_{\mathbb{R} \restriction \alpha} L(\alpha)$  is an ordinal such that  $f(\alpha) \in \mathbb{Q}^*(\aleph_{n+1}^{V[\mathbb{R} \restriction \alpha]}, L(\alpha), V, V[\langle \mathbb{R} \restriction \alpha \rangle])$ .

$\mathbb{R} \restriction \beta$  is partially ordered by  $(p, q, f) \leq (p', q', f')$  if, and only if:

- (1)  $p \leq p'$ ;
- (2)  $\text{dom}(q') \subseteq \text{dom}(q)$ ;
- (3)  $p \restriction \alpha \Vdash_{\mathbb{P} \restriction \alpha} q(\alpha) \leq q'(\alpha)$ , for all  $\alpha \in \text{dom}(q')$ .
- (4)  $\text{dom}(f') \subseteq \text{dom}(f)$ ;
- (5) for all  $\alpha \in \text{dom}(f')$ , if  $(p, q, f) \restriction \alpha := (p \restriction \alpha, q \restriction \alpha, f \restriction \alpha)$ , then

$$(p, q, f) \restriction \alpha \Vdash_{\mathbb{R} \restriction \alpha} f(\alpha) \leq f'(\alpha)$$

Assume that  $V$  is a model of ZFC with two supercompact cardinals  $\kappa < \lambda$ , and  $L : \kappa \rightarrow V_\kappa$  is the Laver function. Let  $\mathbb{R} := \mathbb{R}(\aleph_0, \kappa, L)$  and let  $G_{\mathbb{R}} \subseteq \mathbb{R}$  be any generic filter over  $V$ . Assume that  $G_{\mathbb{M}}$  is an  $\mathbb{M}(\aleph_1, \lambda, V, V[G_{\mathbb{R}}])$ -generic filter over  $V[G_{\mathbb{R}}]$ , we will prove in §8 that both  $\aleph_2$  and  $\aleph_3$  satisfy the super tree property in  $V[G_{\mathbb{R}}][G_{\mathbb{M}}]$ .

## 6. FACTORING MITCHELL'S FORCING

In this section,  $V, W, \eta, \theta$  are like in Definition 5.2. None of the result of this section are due to the author. For more details see [1].

**Remark 6.1.** *The function  $\pi : \mathbb{M}(\eta, \theta, V, W) \rightarrow \mathbb{P}(\eta, \theta)^V$  defined by  $\pi(p, q) := p$  is a projection. If  $\mathbb{P} := \mathbb{P}(\eta, \theta)^V$  and if  $G_{\mathbb{P}}$  is a  $\mathbb{P}$ -generic filter over  $W$ , then we define in  $W[G_{\mathbb{P}}]$  the poset*

$$\mathbb{Q}(\eta, \theta, V, W, G_{\mathbb{P}}) := \mathbb{M}(\eta, \theta, V, W)/G_{\mathbb{P}}.$$

**Lemma 6.2.** *The function  $\sigma : \mathbb{P}(\eta, \theta)^V \times \mathbb{Q}^*(\eta, \theta, V, W) \rightarrow \mathbb{M}(\eta, \theta, V, W)$  defined by  $\sigma(p, (\emptyset, q)) := (p, q)$  is a projection. If  $G_{\mathbb{M}}$  is a  $W$ -generic filter over  $\mathbb{M}(\eta, \theta, V, W)$ , then we define in  $W[G_{\mathbb{M}}]$  the poset:*

$$\mathbb{S}(\eta, \theta, V, W, G_{\mathbb{M}}) := (\mathbb{P}(\eta, \theta)^V \times \mathbb{Q}^*(\eta, \theta, V, W))/G_{\mathbb{M}}.$$

*Proof.* Let  $\mathbb{P} := \mathbb{P}(\eta, \theta)^V$  and  $\mathbb{Q}^* := \mathbb{Q}^*(\eta, \theta, V, W)$ . It is clear that  $\sigma$  preserves the identity and respect the ordering relation. Let  $(p', q') \leq \sigma(p, (\emptyset, q))$ . Define  $q^*$  as follows:  $\text{dom}(q^*) = \text{dom}(q')$  and for  $\alpha \in \text{dom}(q')$ , if  $\alpha \notin \text{dom}(q)$ , then  $q^*(\alpha) := q'(\alpha)$ ; if  $\alpha \in \text{dom}(q)$ , we define  $q^*(\alpha) \in W^{\mathbb{P} \restriction \alpha}$  such that the following hold:

- (1)  $p' \restriction \alpha \Vdash q^*(\alpha) = q'(\alpha)$ ,
- (2) if  $r \in \mathbb{P} \restriction \alpha$  is incompatible with  $p' \restriction \alpha$ , then  $r \Vdash q^*(\alpha) = q(\alpha)$ .

So  $\Vdash_{\mathbb{P} \restriction \alpha}^W q^*(\alpha) \leq q(\alpha)$ , hence  $(p', (\emptyset, q^*)) \leq (p, (\emptyset, q))$  in  $\mathbb{P} \times \mathbb{Q}^*$  and  $\sigma(p', (\emptyset, q^*)) = (p', q^*)$ . Moreover  $[(p', q^*)] = [(p', q')]$ , that completes the proof.  $\square$

**Lemma 6.3.**  $\mathbb{Q}^*(\eta, \theta, V, W)$  is  $\eta^+$ -directed closed in  $W$ .

*Proof.* See [1] for a proof of that lemma.  $\square$

**Lemma 6.4.** *Assume that  $\mathbb{P} := \mathbb{P}(\eta, \theta)^V$  is  $\eta^+$ -cc in  $W$ , for every filter  $G_{\mathbb{M}} \subseteq \mathbb{M}(\eta, \theta, V, W)$  generic over  $W$ , if  $G_{\mathbb{P}} \subseteq \mathbb{P}$  is the projection of  $G_{\mathbb{M}}$  to  $\mathbb{P}$ , then all sets of ordinals in  $W[G_{\mathbb{M}}]$  of size  $\eta$  are in  $W[G_{\mathbb{P}}]$ .*

*Proof.* By Lemma 6.2 it is enough to prove that if  $G_{\mathbb{P}} \times G_{\mathbb{Q}} \subseteq \mathbb{P} \times \mathbb{Q}^*(\eta, \theta, V, W)$  is a generic filter over  $W$ , then every set of ordinals in  $W[G_{\mathbb{P}} \times G_{\mathbb{Q}}]$  of size  $\eta$  is already in  $W[G_{\mathbb{P}}]$ . This is an easy consequence of the Easton's Lemma.  $\square$

**Proposition 6.5.** *Assume that  $\theta$  is inaccessible in  $W$  and let  $\mathbb{M} := \mathbb{M}(\eta, \theta, V, W)$ . The following hold:*

- (i)  $|\mathbb{M}| = \theta$  and  $\mathbb{M}$  is  $\theta$ -c.c.;
- (ii) If  $\mathbb{P}(\eta, \theta)^V$  is  $\eta^+$ -cc in  $W$ , then  $\mathbb{M}$  preserves  $\eta^+$ ;
- (iii) If  $\mathbb{P}(\eta, \theta)^V$  is  $\eta^+$ -c.c. in  $W$ , then  $\mathbb{M}$  makes  $\theta = \eta^{++} = 2^\eta$ .

*Proof.* (i) The proof that  $|\mathbb{M}| = \theta$  is omitted. The key point is that since  $\kappa$  is inaccessible, then  $\mathbb{P}(\eta, \theta)$  has size  $\theta$  and for every  $(p, q) \in \mathbb{M}$ , there are fewer than  $\theta$  possibilities for  $q(\alpha)$ . The proof that  $\mathbb{M}$  is  $\theta$ -c.c. is a standard application of the  $\Delta$ -system Lemma.

(ii) It follows from Lemma 6.4.

(iii) For every cardinal  $\alpha \in ]\eta, \theta[$ ,  $\mathbb{M}$  projects to  $\mathbb{P}(\eta, \alpha)^V$  which makes  $2^\eta \geq \alpha$  and then adds a Cohen subset of  $\eta^+$ . That forcing will collapse  $\alpha$  to  $\eta^+$ . By the previous claims,  $\eta^+$  is preserved and  $\theta$  remains a cardinal after forcing with  $\mathbb{M}$ . So,  $\mathbb{M}$  makes  $\theta = \eta^{++}$ .  $\square$

**Lemma 6.6.** *The following hold:*

(1) *Assume that  $\mathbb{P} := \mathbb{P}(\eta, \theta)^V$ , if  $\mathbb{P}$  adds no new  $< \eta$  sequences to  $W$ , then*

$$\Vdash_{\mathbb{P}}^W \mathbb{Q}(\eta, \theta, V, W, \langle \mathbb{P} \rangle) \text{ is } \eta\text{-closed};$$

(2) *Assume that  $\mathbb{P} := \mathbb{P}(\eta, \theta)^V$  and  $\mathbb{M} := \mathbb{M}(\eta, \theta, V, W)$ . If  $\mathbb{P}$  adds no new  $< \eta$  sequences to  $W$ , then  $\Vdash_{\mathbb{M}}^W \mathbb{S}(\eta, \theta, V, W, \langle \mathbb{M} \rangle)$  is  $\eta$ -closed.*

*Proof.* See [1].  $\square$

For any ordinal  $\alpha \in ]\eta, \theta[$ , the function  $(p, q) \mapsto (p \restriction \alpha, q \restriction \alpha)$  is a projection from  $\mathbb{M}(\eta, \theta, V, W)$  to  $\mathbb{M}_\alpha := \mathbb{M}(\eta, \alpha, V, W)$ . We want to analyse

$$\mathbb{M}(\eta, \theta, V, W)/G_{\mathbb{M}_\alpha},$$

where  $G_{\mathbb{M}_\alpha} \subseteq \mathbb{M}_\alpha$  is any generic filter over  $W$ . Consider the following definition.

**Definition 6.7.** *Let  $\theta' \in ]\eta, \theta[$  be any ordinal and let  $\mathbb{P} := \mathbb{P}(\eta, \theta)^V$ . Let  $\mathbb{M}_{\theta'} := \mathbb{M}(\eta, \theta', V, W)$  and assume that  $G_{\mathbb{M}_{\theta'}} \subseteq \mathbb{M}_{\theta'}$  is any generic filter over  $W$ , then we define in  $W' := W[G_{\mathbb{M}_{\theta'}}]$ , the following poset  $\mathbb{M}(\eta, \theta - \theta', V, W')$ .*

$(p, q) \in \mathbb{M}(\eta, \theta - \theta', V, W')$  if, and only if,

(1)  $p \in \mathbb{P} \restriction (\theta - \theta')$ ;

(2)  $q \in W'$  is a partial function on  $]\theta', \theta[$  of size  $\leq \eta$  such that for every  $\alpha \in \text{dom}(q)$ ,  $\alpha$  is a successor ordinal,  $q(\alpha) \in (W')^{\mathbb{P} \restriction (\alpha - \theta')}$ , and

$$\Vdash_{\mathbb{P} \restriction (\alpha - \theta')}^{W'} q(\alpha) \in \text{Add}(\eta^+)^{W'[\langle \mathbb{P} \restriction (\alpha - \theta') \rangle]}.$$

$\mathbb{M}(\eta, \theta - \theta', V, W)$  is partially ordered as in Definition 5.2.

**Lemma 6.8.** [1, Lemma 2.12] *Let  $\theta' \in ]\eta, \theta[$  be any ordinal and let  $\mathbb{M}_{\theta'} := \mathbb{M}(\eta, \theta', V, W)$  with  $G_{\mathbb{M}_{\theta'}} \subseteq \mathbb{M}_{\theta'}$  a generic filter over  $W$ . Assume that  $\mathbb{P}(\eta, \theta)$  is  $\eta^+$ -cc in  $W$  and in  $W[G_{\mathbb{M}_{\theta'}}]$ , then*

$$\mathbb{M}(\eta, \theta, V, W) \equiv \mathbb{M}_{\theta'} * \mathbb{M}(\eta, \theta - \theta', V, W[\langle \mathbb{M}_{\theta'} \rangle]).$$

*Proof.* One can prove that  $\mathbb{M}_{\theta'} * \mathbb{M}(\eta, \theta - \theta', V, W[\langle \mathbb{M}_{\theta'} \rangle])$  contains a dense set isomorphic to  $\mathbb{M}(\eta, \theta, V, W)$ . The proof is omitted, for more details see [1] Lemma 2.12.  $\square$

**Remark 6.9.** *Lemma 6.2 and Lemma 6.3, can be generalized in the following way. Assume that  $\theta' < \theta$ ,  $\mathbb{P} := \mathbb{P}(\eta, \theta)^V \restriction (\theta - \theta')$ ,  $\mathbb{M}_{\theta'} := \mathbb{M}(\eta, \theta', V, W)$  and  $G_{\mathbb{M}_{\theta'}} \subseteq \mathbb{M}_{\theta'}$  is a generic filter over  $W$ , define*

$$\mathbb{Q}^*(\eta, \theta - \theta', V, W[G_{\mathbb{M}_{\theta'}}]) := \{(\emptyset, q); (\emptyset, q) \in \mathbb{M}(\eta, \theta - \theta', V, W[G_{\mathbb{M}_{\theta'}}])\}.$$

*Then,  $\mathbb{M}(\eta, \theta - \theta', V, W[G_{\mathbb{M}_{\theta'}}])$  is a projection of  $\mathbb{P} \times \mathbb{Q}^*(\eta, \theta - \theta', V, W[G_{\mathbb{M}_{\theta'}}])$  and  $\mathbb{Q}^*(\eta, \theta - \theta', V, W[G_{\mathbb{M}_{\theta'}}])$  is  $\eta^+$ -directed closed in  $W[G_{\mathbb{M}_{\theta'}}]$ .*

## 7. FACTORING THE MAIN FORCING

In this section  $\theta, V, L$  are like in Definition 5.4. We want to analyse the forcing  $\mathbb{R}(\aleph_0, \theta, L)$ . As we said, that poset is a variation of the forcing defined by Abraham in [1, Definition 2.14], we have just to deal with the function  $L$ . The proofs of the lemmas presented in this section are very similar to the proofs of the corresponding lemmas in [1].

**Remark 7.1.**  *$(p, q, f) \mapsto (p, q)$  is a projection of  $\mathbb{R}(\aleph_0, \theta, L)$  to  $\mathbb{M}(\aleph_0, \theta, V, V)$  and for every limit ordinal  $\alpha < \theta$ , if  $L(\alpha)$  is an  $\mathbb{R} \restriction \alpha$ -name for an ordinal and  $\mathbb{Q}^*$  is the canonical  $\mathbb{R} \restriction \alpha$ -name for  $\mathbb{Q}^*(\aleph_1, L(\alpha), V, V[\langle R \restriction \alpha \rangle])$ , then*

$$\mathbb{R} \restriction \alpha + 1 = \mathbb{R} \restriction \alpha * \mathbb{Q}^*.$$

*Indeed, the functions in  $\mathbb{M}(\aleph_0, \theta, V, V)$  are not defined on limit ordinals.*

**Lemma 7.2.** *Let  $\mathbb{U}(\aleph_0, \theta, L) := \{(\emptyset, q, f); (\emptyset, q, f) \in \mathbb{R}\}$  ordered as a subposet of  $\mathbb{R}$ . The following hold:*

- (i) *the function  $\pi : \mathbb{P}(\aleph_0, \theta) \times \mathbb{U}(\aleph_0, \theta, L) \rightarrow \mathbb{R}$  defined by  $\pi(p, (\emptyset, q, f)) = (p, q, f)$  is a projection;*
- (ii)  *$\mathbb{U}(\aleph_0, \theta, L)$  is  $\sigma$ -closed.*

*Proof.* (i) Let  $(p', q', f') \leq \pi(p, (\emptyset, q, f))$ . By Lemma 6.2, the function  $(p, (\emptyset, q)) \mapsto (p, q)$  is a projection and we can find  $(\emptyset, q^*) \leq (\emptyset, q)$  such that  $[(p', q^*)] = [(p', q)]$ . We define a function  $f^*$  as follows:  $\text{dom}(f^*) = \text{dom}(f')$  and for all  $\alpha \in \text{dom}(f')$ , if  $\alpha \notin \text{dom}(f)$ , then  $f^*(\alpha) := f'(\alpha)$ ; if  $\alpha \in \text{dom}(f)$ , we define  $f^*(\alpha) \in V^{\mathbb{R} \restriction \alpha}$  such that the following hold:

- (1)  $(p', q', f') \restriction \alpha \Vdash_{\mathbb{R} \restriction \alpha} f^*(\alpha) = f'(\alpha)$ ,
- (2) if  $r \in \mathbb{R} \restriction \alpha$  is incompatible with  $(p', q', f') \restriction \alpha$ , then  $r \Vdash_{\mathbb{R} \restriction \alpha} f^*(\alpha) = f(\alpha)$ .

Since  $(p', q', f') \restriction \alpha \Vdash_{\mathbb{R} \restriction \alpha} f'(\alpha) \leq f(\alpha)$ , then  $\Vdash_{\mathbb{R} \restriction \alpha} f^*(\alpha) \leq f(\alpha)$ . One can prove by induction on  $\alpha$  that  $[(p^*, q^*, f^*) \restriction \alpha] = [(p', q', f') \restriction \alpha]$ , and we have  $(\emptyset, q^*, f^*) \leq (\emptyset, q, f)$ .

(ii) Let  $\langle (\emptyset, q_n, f_n); n < \omega \rangle$  be a decreasing sequence of conditions in  $\mathbb{U}(\aleph_0, \theta, L)$ . By definition,  $\langle (\emptyset, q_n); n < \omega \rangle$  is a decreasing sequence of conditions in  $\mathbb{Q}^*(\aleph_0, \theta, V, V)$  which is  $\sigma$ -closed by Lemma 6.3. So there is  $(\emptyset, q)$  such that  $(\emptyset, q) \leq (\emptyset, q_n)$ , for every

$n < \omega$ . We define a function  $f$  with  $\text{dom}(f) = \bigcup_{n < \omega} \text{dom}(f_n)$  as follows. We define  $f \restriction \alpha + 1$  by induction on  $\alpha$ , so that

$$(\emptyset, q \restriction \alpha + 1, f \restriction \alpha + 1) \leq (\emptyset, q_n, f_n) \restriction \alpha + 1,$$

for all  $n < \omega$ . Assume  $f \restriction \alpha$  has been defined. For every  $m > n$ , we have

$$(\emptyset, q_m, f_m) \restriction \alpha \Vdash_{\mathbb{R} \restriction \alpha} f_m(\alpha) \leq f_n(\alpha),$$

so by the inductive hypothesis we have  $(\emptyset, q \restriction \alpha, f \restriction \alpha) \Vdash f_m(\alpha) \leq f_n(\alpha)$ . By Lemma 6.3, if  $G_\alpha \subseteq \mathbb{R} \restriction \alpha$  is a generic filter over  $V$ , then  $\mathbb{Q}^*(\aleph_1, L(\alpha), V, V[G_\alpha])$  is  $\aleph_2$ -closed in  $V[G_\alpha]$ . It follows that for some  $f(\alpha) \in V^{\mathbb{R} \restriction \alpha}$ , we have

$$(\emptyset, q \restriction \alpha, f \restriction \alpha) \Vdash f(\alpha) \leq f_m(\alpha), \text{ for every } m < \omega.$$

Finally, the condition  $(\emptyset, q, f)$  is a lower bound for  $\langle (\emptyset, q_n, f_n); n < \omega \rangle$ .  $\square$

**Lemma 7.3.** *Assume that  $V$  is a model of ZFC with two supercompact cardinals  $\kappa < \lambda$ , and  $L : \kappa \rightarrow V_\kappa$  is the Laver function. Let  $\mathbb{R} := \mathbb{R}(\aleph_0, \kappa, L)$ , and let  $\dot{\mathbb{M}}$  be the canonical  $\mathbb{R}$ -name for  $\mathbb{M}(\aleph_1, \lambda, V, V[\langle \mathbb{R} \rangle])$ . The following hold:*

- (1)  $\mathbb{R}$  has size  $\kappa$  and it is  $\kappa$ -c.c.;
- (2)  $\Vdash_{\mathbb{R}} \lambda$  is inaccessible;
- (3) For every filter  $G_{\mathbb{R}} \subseteq \mathbb{R}$  generic over  $V$ , if  $G_0$  is the projection of  $G_{\mathbb{R}}$  to  $\mathbb{P}_0 := \mathbb{P}(\aleph_0, \kappa)$ , then all countable sets of ordinals in  $V[G_{\mathbb{R}}]$  are in  $V[G_0]$ ;
- (4)  $\mathbb{R}$  preserves  $\aleph_1$  and makes  $\kappa = \aleph_2 = 2^{\aleph_0}$ ;
- (5) If  $G_{\mathbb{R}} \subseteq \mathbb{R}$  is a generic filter over  $V$ , then  $\mathbb{P}_1 := \mathbb{P}(\aleph_1, \lambda)^V$  does not introduce new countable subsets to  $V[G_{\mathbb{R}}]$ ;
- (6)  $\Vdash_{\mathbb{R}} \mathbb{P}(\aleph_1, \lambda)^V$  is  $\kappa$ -c.c. (and even  $\kappa$ -Knaster).

*Proof.* (1) The proof is similar to the proof of Lemma 6.5 (i) and it is omitted.

(2) It follows from the previous claim.

(3) By Lemma 7.2, it is enough to prove that if  $G_0 \times H \subseteq \mathbb{P}_0 \times \mathbb{U}(\aleph_0, \kappa, L)$  is any generic filter over  $V$ , then every countable set of ordinals in  $V[G_0 \times H]$  is already in  $V[G_0]$ . This is an easy consequence of Easton's Lemma.

(4) Since  $\mathbb{P}(\aleph_0, \kappa)$  is c.c.c., Claim 3 implies that  $\aleph_1$  is preserved. Since  $\mathbb{R}$  is  $\kappa$ -c.c., then  $\kappa$  remains a cardinal after forcing with  $\mathbb{R}$ . Moreover,  $\mathbb{R}$  projects on  $\mathbb{M}(\aleph_0, \kappa, V, V)$  that, by Proposition 6.5, collapses all the cardinals between  $\aleph_1$  and  $\kappa$  and adds  $\kappa$  many Cohen reals. Therefore  $\mathbb{R}$  makes  $\kappa = \aleph_2 = 2^{\aleph_0}$ .

(5) By Lemma 7.2,  $\mathbb{R}$  is a projection of  $\mathbb{P}_0 \times \mathbb{U}_0$ , where  $\mathbb{P}_0 := \mathbb{P}(\aleph_0, \kappa)$  and  $\mathbb{U} := \mathbb{U}(\aleph_0, \kappa, L)$ . By Easton's Lemma  $\Vdash_{\mathbb{P}_0 \times \mathbb{U}} \mathbb{P}_1$  is  $< \aleph_1$ -distributive, so no countable sequence of ordinals is added by  $\mathbb{P}_1$  to  $V[G_0 \times H]$ , where  $G_0 \subseteq \mathbb{P}_0$  and  $H \subseteq \mathbb{U}$  are generic filters over  $V$  such that  $G_{\mathbb{R}}$  is the projection of  $G_0 \times H$  to  $\mathbb{R}$ . Moreover, we proved in Claim 3, that every countable sequence of ordinals in  $V[G_0 \times H]$  is already in  $V[G_0]$ . Since  $V[G_0] \subseteq V[G_{\mathbb{R}}]$ , this completes the proof.

(6) Let  $G_{\mathbb{R}} \subseteq \mathbb{R}$  be a generic filter over  $V$ . Work in  $V[G_{\mathbb{R}}]$ . Assume that  $\langle f_\alpha; \alpha < \kappa \rangle$

is a sequence of conditions in  $\mathbb{P}_1 := \mathbb{P}(\aleph_1, \lambda)^V$ . Let  $D := \bigcup_{\alpha < \kappa} \text{dom}(f_\alpha)$ , then there is a bijection  $h : D \rightarrow \kappa$ . Since every condition of the sequence is a countable function we have, for every  $\alpha < \kappa$  of uncountable cofinality  $\sup(h''[\text{dom}(f_\alpha)] \cap \alpha) < \alpha$ . So the function  $\alpha \mapsto \sup(h''[\text{dom}(f_\alpha)] \cap \alpha)$  is regressive. By Fodor's Theorem, there is an ordinal  $\tau$  and a stationary set  $S \subseteq \kappa$  such that  $\sup(h''[\text{dom}(f_\alpha)] \cap \alpha) = \tau$ , for every  $\alpha \in S$ . The set  $h^{-1}(\tau)$  has size  $< \kappa$  in  $V[G_{\mathbb{R}}]$  and  $\mathbb{R}$  is  $\kappa$ -c.c., so there is a set  $E \in V$  of size  $< \kappa$  in  $V$  such that  $h^{-1}(\tau) \subseteq E$ . Since  $\kappa$  is inaccessible in  $V$ , then we can find in  $V[G_{\mathbb{R}}]$  a stationary set  $S' \subseteq S$  such that  $f_\alpha \upharpoonright E$  has a fixed value, for every  $\alpha \in S'$ . Then the sets in  $\{\text{dom}(f_\alpha) \setminus E; \alpha \in S'\}$  can be assumed to be pairwise disjoint, hence  $f_\alpha \cup f_\beta$  is a function for every  $\alpha, \beta \in S'$ .  $\square$

**Lemma 7.4.** [1, Lemma 2.18] *Assume that  $\alpha < \theta$  is a limit ordinal, let  $\mathbb{P} := \mathbb{P}(\aleph_0, \theta) \upharpoonright (\theta - \alpha)$ ,  $\mathbb{R} := \mathbb{R}(\aleph_0, \theta, L)$  and let  $G_\alpha \subseteq \mathbb{R} \upharpoonright \alpha + 1$  be a generic filter over  $V$ . We define in  $V[G_{\alpha+1}]$  the following set:*

$$\mathbb{U}_{\alpha+1}(\aleph_0, \theta, L, G_{\alpha+1}) := \{(0, q, f) \in \mathbb{R}(\aleph_0, \theta, L); (0, q, f) \upharpoonright \alpha + 1 \in G_{\alpha+1}\}.$$

*Then  $\mathbb{R}/G_{\alpha+1}$  is a projection of  $\mathbb{P} \times \mathbb{U}_{\alpha+1}(\aleph_0, \theta, L, G_{\alpha+1})$ , and  $\mathbb{U}_{\alpha+1}(\aleph_0, \theta, L, G_{\alpha+1})$  is  $\sigma$ -closed in  $V[G_{\alpha+1}]$ .*

*Proof.* The proof is very similar to the proof of Lemma 2.18 in [1] and it is omitted.  $\square$

## 8. THE MAIN THEOREM

**Theorem 8.1.** *Assume that  $V$  is a model of ZFC with two supercompact cardinals  $\kappa < \lambda$ , and suppose that  $L : \kappa \rightarrow V_\kappa$  is the Laver function. If  $\mathbb{R} := \mathbb{R}(\aleph_0, \kappa, L)$ , and  $\dot{\mathbb{M}}$  is the canonical  $\mathbb{R}$ -name for  $\mathbb{M}(\aleph_1, \lambda, V, V[\langle \mathbb{R} \rangle])$ , then for every filter  $G \subseteq \mathbb{R} * \dot{\mathbb{M}}$  generic over  $V$ , both  $\aleph_2$  and  $\aleph_3$  satisfy the super tree property in  $V[G]$ .*

The proof that the model obtained is as required consists of three parts:

- (1)  $V[G] \models \aleph_1^V = \aleph_1$ ,  $\kappa = \aleph_2 = 2^{\aleph_0}$  and  $\lambda = \aleph_3 = 2^{\aleph_1}$ ;
- (2)  $\aleph_3$  satisfies the super tree property.
- (3)  $\aleph_2$  satisfies the super tree property;

### PROOF OF (1)

First we show that  $\aleph_1$  is preserved. Let  $G_{\mathbb{R}}$  be the projection of  $G$  to  $\mathbb{R}$  and let  $G_{\dot{\mathbb{M}}}$  be the projection of  $G$  to  $\dot{\mathbb{M}} := \dot{\mathbb{M}}^{G_{\mathbb{R}}}$ . By Lemma 7.3,  $\aleph_1$  is preserved by  $\mathbb{R}$ . Moreover,  $\mathbb{P}(\aleph_1, \lambda)^V$  does not introduce new countable subsets to  $V[G_{\mathbb{R}}]$  (see Lemma 7.3 (5)). So, by Lemma 6.6 (1)  $\dot{\mathbb{M}}$  does not introduce new countable sequences, hence  $\aleph_1$  remains a cardinal in  $V[G]$ . Now, we show that  $\kappa$  remains a cardinal in  $V[G]$ . By Lemma 7.3, we know that  $\kappa$  remains a cardinal in  $V[G_{\mathbb{R}}]$  and becomes  $\aleph_2$ . By Lemma 7.3 (6),  $\mathbb{P}(\aleph_1, \lambda)^V$  is  $\kappa$ -c.c. in  $V[G_{\mathbb{R}}]$ , so  $\kappa$  remains a cardinal after forcing with  $\mathbb{P}(\aleph_1, \lambda)^V$  over  $V[G_{\mathbb{R}}]$  and it is equal to  $\aleph_2$ . By applying Lemma 6.4, we get that all sets of ordinals in  $V[G]$  of cardinality  $\aleph_1$  are in  $V[G_{\mathbb{R}}][G_{\mathbb{P}}]$ , where  $G_{\mathbb{P}}$  is the projection of  $G_{\dot{\mathbb{M}}}$  to  $\mathbb{P}(\aleph_1, \lambda)^V$ .

Therefore,  $\kappa$  remains a cardinal in  $V[G]$ . Finally,  $\lambda$  remains a cardinal because  $\mathbb{R} * \dot{\mathbb{M}}$  is  $\lambda$ -c.c., and it becomes  $\aleph_3$ .

## PROOF OF 2

By (1), we know that  $\lambda = \aleph_3$  in  $V[G]$ , so we want to prove that  $\lambda$  has the super tree property in that model. Let  $\mu \geq \lambda$  be any ordinal, and assume towards a contradiction that in  $V[G]$  there is a  $(\lambda, \mu)$ -tree  $F$  and an  $F$ -level sequence  $D$  with no ineffable branches. Fix an elementary embedding  $j : V \rightarrow N$  with critical point  $\lambda$  such that:

- (1) if  $\sigma := |\mu|^{<\lambda}$ , then  $j(\lambda) > \sigma$ ,
- (2)  ${}^\sigma N \subseteq N$ .

**Claim 8.2.** *We can lift  $j$  to an elementary embedding  $j^* : V[G] \rightarrow N[H]$ , with  $H \subseteq j(\mathbb{R} * \dot{\mathbb{M}})$  generic over  $N$ .*

*Proof.* To simplify the notation we will denote all the extensions of  $j$  by “ $j$ ” also. We let  $G_{\mathbb{R}}$  be the projection of  $G$  to  $\mathbb{R}$  and let  $G_{\mathbb{M}}$  be the projection of  $G$  to  $\mathbb{M} := \dot{\mathbb{M}}^{G_{\mathbb{R}}}$ .

As  $\lambda > \kappa$  and  $|\mathbb{R}| = \kappa$ , we have  $j(\mathbb{R}) = \mathbb{R}$ , so we can lift  $j$  to an elementary embedding

$$j : V[G_{\mathbb{R}}] \rightarrow N[G_{\mathbb{R}}].$$

Observe that  $j(\mathbb{M}) \restriction \lambda = \mathbb{M}(\aleph_1, \lambda, N, N[G_{\mathbb{R}}]) = \mathbb{M}(\aleph_1, \lambda, V, V[G_{\mathbb{R}}]) = \mathbb{M}$ . Force over  $V[G_{\mathbb{R}}]$  to get a  $j(\mathbb{M})$ -generic filter  $H_{j(\mathbb{M})}$  such that  $H_{j(\mathbb{M})} \restriction \lambda = G_{\mathbb{M}}$ . By Lemma 6.5 and Lemma 7.3 (2),  $\mathbb{M}$  is  $\lambda$ -c.c. in  $V[G_{\mathbb{R}}]$ , so  $j \restriction \mathbb{M}$  is a complete embedding from  $\mathbb{M}$  into  $j(\mathbb{M})$ , hence we can lift  $j$  to an elementary embedding

$$j : V[G_{\mathbb{R}}][G_{\mathbb{M}}] \rightarrow N[G_{\mathbb{R}}][H_{j(\mathbb{M})}].$$

□

Rename  $j^*$  by  $j$ . We define  $\mathcal{N}_1 := N[G]$  and  $\mathcal{N}_2 := N[G_{\mathbb{R}}][H_{j(\mathbb{M})}]$ . In  $\mathcal{N}_2$ ,  $j(F)$  is a  $(j(\lambda), j(\mu))$ -tree and  $j(D)$  is a  $j(F)$ -level sequence. By the closure of  $N$ , the tree  $F$  and the  $F$ -level sequence  $D$  are in  $\mathcal{N}_1$ , and there is no ineffable branch for  $D$  in  $\mathcal{N}_1$ .

**Claim 8.3.** *In  $\mathcal{N}_2$ , there is an ineffable branch  $b$  for  $D$ .*

*Proof.* Let  $a := j''[\mu]$ , clearly  $a \in [j(\mu)]^{<j(\lambda)}$ . Consider  $f := j(D)(a)$ , let  $b : \mu \rightarrow 2$  be the function defined by  $b(\alpha) := f(j(\alpha))$ , we show that  $b$  is an ineffable branch for  $D$ . Assume for a contradiction that in  $\mathcal{N}_2$  there is a club  $C \subseteq [\mu]^{<|\lambda|} \cap \mathcal{N}_1$  such that  $b \restriction X \neq D(X)$ , for all  $X \in C$ . Then by elementarity,

$$f \restriction X \neq j(D)(X),$$

for all  $X \in j(C)$ . But  $a \in j(C)$  and  $f = j(D)(a)$ , so we have a contradiction. □

Since there is no ineffable branch for  $D$  in  $\mathcal{N}_1$ , we get a contradiction by proving that forcing with  $\mathbb{M}(\aleph_1, j(\lambda) - \lambda, N, N[G])$  over  $N[G]$  does not add ineffable branches to  $D$ . By Remark 6.9,  $\mathbb{M}(\aleph_1, j(\lambda) - \lambda, N, N[G])$  is a projection of

$$\mathbb{P} \times \mathbb{Q}^*(\aleph_1, j(\lambda) - \lambda, N, N[G]),$$

where  $\mathbb{P} := \mathbb{P}(\aleph_1, j(\lambda))^N \restriction (j(\lambda) - \lambda)$ , and  $\mathbb{Q}^* := \mathbb{Q}^*(\aleph_1, j(\lambda) - \lambda, N, N[G])$  is  $\aleph_2$ -closed in  $N[G]$ . In  $N[G]$ , we have  $\lambda = \aleph_3 = 2^{\aleph_1}$  and  $F$  is an  $(\aleph_3, \mu)$ -tree, so we can apply Corollary 4.2, thus  $\mathbb{Q}^*$  cannot add ineffable branches to  $D$ .

This means that  $b \notin N[G][H_{\mathbb{Q}^*}]$ , where  $H_{\mathbb{Q}^*}$  is the projection of  $H_{j(\mathbb{M})}$  to  $\mathbb{Q}^*$ . The filter  $H_{\mathbb{Q}^*}$  collapses  $\lambda$  (which is  $\aleph_3^{N[G]}$ ) to have size  $\aleph_2$ , so now  $F$  is an  $(\aleph_2, \mu)$ -tree. Every set  $X \subseteq N$  in  $N[G][H_{\mathbb{Q}^*}]$  which has size  $< \aleph_2$  in  $N[G][H_{\mathbb{Q}^*}]$  is covered by a set  $Y \in N$  which has size  $< \lambda$  in  $N$ . The model  $N[G][H_{j(\mathbb{M})}]$  is the result of forcing with  $\mathbb{P}$  over  $N[G][H_{\mathbb{Q}^*}]$  and  $b$  is in  $N[G][H_{j(\mathbb{M})}]$ , so by Theorem 4.6,  $b \in N[G][H_{\mathbb{Q}^*}]$ , a contradiction.

This completes the proof of (2).

### PROOF OF 3

By (1), we know that  $\kappa = \aleph_2$  in  $V[G]$ , so we want to prove that  $\kappa$  has the super tree property in that model. Let  $\mu \geq \kappa$  be any ordinal, and assume towards a contradiction that in  $V[G]$  there is a  $(\kappa, \mu)$ -tree  $F$  and an  $F$ -level sequence  $D$  with no ineffable branches. Since  $L$  is the Laver function, there is an elementary embedding  $j : V \rightarrow N$  with critical point  $\kappa$  such that:

- (1) if  $\sigma := \max(\lambda, |\mu|^{<\kappa})$ , then  $j(\kappa) > \sigma$ ,
- (2)  ${}^\sigma N \subseteq N$ ,
- (3)  $j(L)(\kappa) = \lambda$ .

**Claim 8.4.** *We can lift  $j$  to an elementary embedding  $j^* : V[G] \rightarrow N[H]$ , with  $H \subseteq j(\mathbb{R} * \mathbb{M})$  generic over  $N$ .*

*Proof.* To simplify the notation we will denote all the extensions of  $j$  by “ $j$ ” also. Let  $G_{\mathbb{R}}$  be the projection of  $G$  to  $\mathbb{R}$  and let  $G_{\mathbb{M}}$  be the projection of  $G$  to  $\mathbb{M} := \dot{\mathbb{M}}^{G_{\mathbb{R}}}$ . Observe that  $j(\mathbb{R}) = \mathbb{R}(\aleph_0, j(\kappa), j(L))^N = \mathbb{R}(\aleph_0, j(\kappa), j(L))^V$ , and  $j(\mathbb{R}) \restriction \kappa = \mathbb{R}$ . Force over  $V$  to get a  $j(\mathbb{R})$ -generic filter  $H_{j(\mathbb{R})}$  such that  $H_{j(\mathbb{R})} \restriction \kappa = G_{\mathbb{R}}$ . By Lemma 7.3 (1)  $\mathbb{R}$  is  $\kappa$ -c.c. So  $j \restriction \mathbb{R}$  is a complete embedding from  $\mathbb{R}$  into  $j(\mathbb{R})$ , hence we can lift  $j$  to get an elementary embedding

$$j : V[G_{\mathbb{R}}] \rightarrow N[H_{j(\mathbb{R})}].$$

By Lemma 6.2, in  $V[G_{\mathbb{R}}]$ , the forcing  $\mathbb{M}$  is a projection of

$$\mathbb{P}(\aleph_1, \lambda)^V \times \mathbb{Q}^*(\aleph_1, \lambda, V, V[G_{\mathbb{R}}])$$

(moreover,  $\mathbb{P}(\aleph_1, \lambda)^V = \mathbb{P}(\aleph_1, \lambda)^N$  and  $\mathbb{Q}^*(\aleph_1, \lambda, V, V[G_{\mathbb{R}}]) = \mathbb{Q}^*(\aleph_1, \lambda, N, N[G_{\mathbb{R}}])$ ). Recall that

$$\mathbb{S}(\aleph_1, \lambda, V, V[G_{\mathbb{R}}], G_{\mathbb{M}}) = (\mathbb{P}(\aleph_1, \lambda)^V \times \mathbb{Q}^*(\aleph_1, \lambda, V, V[G_{\mathbb{R}}]))/G_{\mathbb{M}},$$

so by forcing with  $\mathbb{S}(\aleph_1, \lambda, V, V[G_{\mathbb{R}}], G_{\mathbb{M}})$  over  $V[G]$  we obtain a model  $V[G_{\mathbb{R}}][G_{\mathbb{P}} \times G_{\mathbb{Q}^*}]$  with  $G_{\mathbb{P}} \times G_{\mathbb{Q}^*}$  generic for  $\mathbb{P}(\aleph_1, \lambda)^V \times \mathbb{Q}^*(\aleph_1, \lambda, V, V[G_{\mathbb{R}}])$  over  $V[G_{\mathbb{R}}]$  and such that



$G_{\mathbb{M}}$  is the projection of  $G_{\mathbb{P}} \times G_{\mathbb{Q}^*}$  to  $\mathbb{M}$ .

If  $\mathbb{P} := \mathbb{P}(\aleph_1, \lambda)^V$ , then  $\mathbb{P}$  is  $\kappa$ -c.c. in  $V[G_{\mathbb{R}}]$  (Lemma 7.3 (6)), hence  $j \restriction \mathbb{P}$  is a complete embedding of  $\mathbb{P}$  into  $j(\mathbb{P})$ . Moreover,  $\mathbb{P}$  is isomorphic via  $j \restriction \mathbb{P}$  to  $\mathbb{P}(\aleph_1, j''[\lambda])^N = \mathbb{P}(\aleph_1, j''[\lambda])^V$ . By forcing with  $\mathbb{P}(\aleph_1, j(\lambda))^V \restriction (j(\lambda) - j''[\lambda])$  over  $V[H_{j(\mathbb{R})}]$  we get a  $j(\mathbb{P})$ -generic filter  $H_{j(\mathbb{P})}$  such that  $j''[G_{\mathbb{P}}] \subseteq H_{j(\mathbb{P})}$ . Then  $j$  lifts to an elementary embedding

$$j : V[G_{\mathbb{R}}][G_{\mathbb{P}}] \rightarrow N[H_{j(\mathbb{R})}][H_{j(\mathbb{P})}].$$

Let  $\mathbb{Q}^* := \mathbb{Q}^*(\aleph_1, \lambda, V, V[G_{\mathbb{R}}])$ . By Remark 7.1 and since  $j(\mathbb{R}) \restriction \kappa = \mathbb{R}$ , we have  $j(\mathbb{R}) \restriction \kappa + 1 = \mathbb{R} * \dot{\mathbb{Q}}^*$  where  $\dot{\mathbb{Q}}^*$  is an  $\mathbb{R}$ -name for  $\mathbb{Q}^*(\aleph_1, j(L)(\kappa), V, V[G_{\mathbb{R}}])$ . We chose  $j$  so that  $j(L)(\kappa) = \lambda$ , therefore forcing with  $j(\mathbb{R}) \restriction \kappa + 1$  over  $V$  is the same as forcing with  $\mathbb{R}$  followed by forcing with  $\mathbb{Q}^*$  over  $V[G_{\mathbb{R}}]$ . It follows that, by the closure of  $N$ , we have  $j''[G_{\mathbb{Q}^*}] \in N[H_{j(\mathbb{R})}]$ . By Lemma 6.3,  $\mathbb{Q}^*$  is  $\aleph_2$ -directed closed in  $V[G_{\mathbb{R}}]$ , hence  $j(\mathbb{Q}^*)$  is  $\aleph_2$ -directed closed in  $N[H_{j(\mathbb{R})}]$ . Moreover, the filter  $H_{j(\mathbb{R})}$  collapses  $\lambda$  to have size  $\aleph_1$ , thus  $j''[G_{\mathbb{Q}^*}]$  has size  $\aleph_1$  in  $V[H_{j(\mathbb{R})}]$ . Therefore, we can find  $t \leq j(q)$ , for all  $q \in G_{\mathbb{Q}^*}$ . We force over  $V[G_{j(\mathbb{R})}]$  with  $j(\mathbb{Q}^*)$  below  $t$  to get a  $j(\mathbb{Q}^*)$ -generic filter  $H_{j(\mathbb{Q}^*)}$  containing  $j''[G_{\mathbb{Q}^*}]$ . The filter  $H_{j(\mathbb{P})} \times H_{j(\mathbb{Q}^*)}$  generates a filter  $H_{j(\mathbb{M})}$  generic for  $j(\mathbb{M})$  over  $N[H_{j(\mathbb{R})}]$ .

It remains to prove that  $j''[G_{\mathbb{M}}] \subseteq H_{j(\mathbb{M})}$ : let  $(p, q)$  be a condition of  $G_{\mathbb{M}}$ , there are  $\bar{p} \in G_{\mathbb{P}}$  and  $(0, \bar{q}) \in G_{\mathbb{Q}^*}$  such that  $(\bar{p}, \bar{q}) \leq (p, q)$ . We have  $j(\bar{p}) \in H_{j(\mathbb{P})}$  and  $(0, j(\bar{q})) \in H_{j(\mathbb{Q}^*)}$ , hence  $(j(\bar{p}), 0)$  and  $(0, j(\bar{q}))$  are both in  $H_{j(\mathbb{M})}$ . The condition  $j(\bar{p}, \bar{q})$  is the greatest lower bound<sup>1</sup> of  $(j(\bar{p}), 0)$  and  $(0, j(\bar{q}))$ ; it follows that  $j(\bar{p}, \bar{q}) \in H_{j(\mathbb{M})}$ . We also have  $j(\bar{p}, \bar{q}) \leq j(p, q)$ , hence  $j(p, q) \in H_{j(\mathbb{M})}$  as required. Therefore,  $j$  lifts to an elementary embedding

$$j : V[G_{\mathbb{R}}][G_{\mathbb{M}}] \rightarrow N[H_{j(\mathbb{R})}][H_{j(\mathbb{M})}].$$

□

Rename  $j^*$  by  $j$ . We define  $\mathcal{N}_1 := N[G]$  and  $\mathcal{N}_2 := N[H_{j(\mathbb{R})}][H_{j(\mathbb{M})}]$ . In  $\mathcal{N}_2$ ,  $j(F)$  is a  $(j(\kappa), j(\mu))$ -tree and  $j(D)$  is a  $j(F)$ -level sequence. By the closure of  $N$ , the tree  $F$  and the  $F$ -level sequence  $D$  are in  $\mathcal{N}_1$ , and there is no ineffable branch for  $D$  in  $\mathcal{N}_1$ .

**Claim 8.5.** *In  $\mathcal{N}_2$ , there is an ineffable branch  $b$  for  $D$ .*

*Proof.* Let  $a := j''[\mu]$ , clearly  $a \in [j(\mu)]^{<j(\kappa)}$ . Consider  $f := j(D)(a)$ , let  $b : \mu \rightarrow 2$  be the function defined by  $b(\alpha) := f(j(\alpha))$ , we show that  $b$  is an ineffable branch for  $D$ . Assume for a contradiction that for some club  $C \subseteq [\mu]^{<|\kappa|} \cap \mathcal{N}_1$  we have  $b \restriction X \neq D(X)$ , for all  $X \in C$ . Then by elementarity,

$$f \restriction X \neq j(D)(X),$$

<sup>1</sup>  $j(\bar{p}, \bar{q}) = (j(\bar{p}), j(\bar{q}))$  is clearly a lower bound. Suppose that  $(p_1, q_1)$  is also a lower bound, then by definition  $p_1 \leq j(\bar{p})$  and  $p_1 \restriction \alpha \Vdash q_1(\alpha) \leq j(\bar{q})(\alpha)$ , for every  $\alpha$ . That is  $(p_1, q_1) \leq (j(\bar{p}), j(\bar{q}))$ .

for all  $X \in j(C)$ . But  $a \in j(C)$  and  $f = j(D)(a)$ , so we have a contradiction.  $\square$

Since there is no ineffable branch for  $D$  in  $\mathcal{N}_1$ , we get a contradiction with the following claim.

**Claim 8.6.**  $b \in \mathcal{N}_1$ .

*Proof.* Assume towards a contradiction that  $b \notin \mathcal{N}_1$ . By Lemma 7.3 (5) and Lemma 6.6 (2), the poset  $\mathbb{S} := \mathbb{S}(\aleph_1, \lambda, N, N[G_{\mathbb{R}}], G_{\mathbb{M}})$  is  $\sigma$ -closed in  $\mathcal{N}_1$ . In  $\mathcal{N}_1$ , we have  $\kappa = \aleph_2 = 2^{\aleph_0}$ , hence  $F$  is a  $(\aleph_2, \mu)$ -tree and we can apply the First Preservation Theorem to  $\mathbb{S}$ , thus

$$b \notin N[G_{\mathbb{R}}][G_{\mathbb{P}} \times G_{\mathbb{Q}^*}]$$

(we defined  $G_{\mathbb{P}} \times G_{\mathbb{Q}^*}$  in Claim 8.4 as a generic filter for  $\mathbb{S}$ ).  $\mathbb{S}$  is  $< \aleph_2$ -distributive in  $\mathcal{N}_1$  (this is a standard application of the Easton's Lemma, see [2, Lemma 3.20] for more details) so  $F$  is still an  $(\aleph_2, \mu)$ -tree after forcing with  $\mathbb{S}$ . Now, the forcing that takes us from  $\mathbb{P}$  to  $j(\mathbb{P})$  is

$$\mathbb{P}_{tail} := \mathbb{P}(\aleph_1, j(\lambda))^N \upharpoonright (j(\lambda) - \lambda).$$

The pair  $(N, N[G_{\mathbb{R}}][G_{\mathbb{P}} \times G_{\mathbb{Q}^*}])$  has the  $\kappa$ -covering property, because  $\mathbb{S}$  is  $< \aleph_2$ -distributive and  $\mathbb{R}$  is  $\kappa$ -c.c. Since  $\kappa$  is inaccessible in  $N$ , we can apply the Second Preservation Theorem to  $\mathbb{P}_{tail}$ , so

$$b \notin N[G_{\mathbb{R}}][G_{\mathbb{Q}^*}][H_{j(\mathbb{P})}].$$

We already observed in the proof of the first claim that forcing with  $j(\mathbb{R}) \upharpoonright \kappa + 1$  over  $V$  is the same as forcing with  $\mathbb{R}$  followed by forcing with  $\mathbb{Q}^*$  over  $V[G_{\mathbb{R}}]$ . So, if  $H_{\kappa+1}$  is the projection of  $H_{j(\mathbb{R})}$  to  $j(\mathbb{R}) \upharpoonright \kappa + 1$ , then  $N[G_{\mathbb{R}}][G_{\mathbb{Q}^*}] = N[H_{\kappa+1}]$ . This means that we proved

$$b \notin N[H_{\kappa+1}][H_{j(\mathbb{P})}].$$

Consider  $\mathbb{R}_{tail} := j(\mathbb{R})/H_{\kappa+1}$ , by Lemma 7.4,  $\mathbb{R}_{tail}$  is a projection of  $\mathbb{P}_0 \times \mathbb{U}_0$ , where  $\mathbb{P}_0 := \mathbb{P}(\aleph_0, j(\kappa))^N \upharpoonright (j(\kappa) - \kappa)$  and  $\mathbb{U}_0 := \mathbb{U}_{\kappa+1}(\aleph_0, j(\kappa), j(L), H_{\kappa+1})$ , moreover,  $\mathbb{U}_0$  is  $\sigma$ -closed in  $N[H_{\kappa+1}]$ . Forcing with  $j(\mathbb{P})$  does not add countable sequences to  $N[H_{\kappa+1}]$  (the proof is analogous to the proof of Lemma 7.3 (5)) hence  $\mathbb{U}_0$  is still  $\sigma$ -closed in  $N[H_{\kappa+1}][H_{j(\mathbb{P})}]$ .

We want to apply the First Preservation Theorem to  $\mathbb{U}_0$ , so consider the following facts. In  $N[H_{\kappa+1}][H_{j(\mathbb{P})}]$ , we have  $2^{\aleph_0} = j(\kappa) > \kappa = \aleph_2$  but now  $F$  is not exactly an  $(\aleph_2, \mu)$ -tree because  $N[H_{\kappa+1}][H_{j(\mathbb{P})}]$  was obtained by forcing with  $\mathbb{P}_{tail}$  over  $N[G_{\mathbb{R}}][G_{\mathbb{P}} \times G_{\mathbb{Q}^*}]$  and  $\mathbb{P}_{tail}$  is not  $< \aleph_2$ -distributive. Yet,  $\mathbb{P}_{tail}$  is  $\aleph_2$ -c.c. in  $N[G_{\mathbb{R}}][G_{\mathbb{P}} \times G_{\mathbb{Q}^*}]$  (the proof is analogous to the proof of Lemma 7.3 (6), see [2] for more details), so  $F$  "covers" an  $(\aleph_2, \mu)$ -tree, namely there is in  $N[H_{\kappa+1}][H_{j(\mathbb{P})}]$  a  $(\aleph_2, \mu)$ -tree  $F^*$  such that for cofinally many  $X \in [\mu]^{<\aleph_2}$ ,  $\text{Lev}_X(F) \subseteq \text{Lev}_X(F^*)$ . Let  $N[H_{\mathbb{U}_0}][H_{j(\mathbb{P})}]$  be the generic extension obtained by forcing with  $\mathbb{U}_0$  over  $N[H_{\kappa+1}][H_{j(\mathbb{P})}]$ . If  $b \in N[H_{\mathbb{U}_0}][H_{j(\mathbb{P})}]$ , then  $b$  provides a cofinal branch for  $F^*$  in that model, hence by the First Preservation

Theorem,  $b \in N[H_{\kappa+1}][H_{j(\mathbb{P})}]$ . But we already proved that  $b$  does not belong to that model, so we must have

$$b \notin N[H_{\mathbb{U}_0}][H_{j(\mathbb{P})}].$$

The filter  $H_{\mathbb{U}_0}$  collapses  $\kappa$  (hence  $\aleph_2$ ) to have size  $\aleph_1$ , so now  $F^*$  is an  $(\aleph_1, \mu)$ -tree in  $W := N[H_{\mathbb{U}_0}][H_{j(\mathbb{P})}]$ . The model  $N[H_{j(\mathbb{R})}][H_{j(\mathbb{P})}]$  is the result of forcing with  $\mathbb{P}_0$  over  $W$ . Observe that  $\mathbb{P}_0$  and  $(W, W)$  satisfy all the hypothesis of the Second Preservation Theorem: indeed,  $\mathbb{P}_0 \subseteq \text{Add}(\aleph_0, j(\kappa))^W$  and in  $W$ , we have  $\gamma^{<\omega} < \omega_1$  for every cardinal  $\gamma < \omega_1$ . Therefore,

$$b \notin N[H_{j(\mathbb{R})}][H_{j(\mathbb{P})}].$$

$\mathbb{P}_0$  is c.c.c. in  $W$  (the proof is analogous to the proof of Lemma 7.3 (6)), so  $F^*$  covers an  $(\aleph_1, \mu)$ -tree in  $N[H_{j(\mathbb{R})}][H_{j(\mathbb{P})}]$ , we rename it  $F^*$ .  $\mathcal{N}_2$  is the result of forcing with

$$\mathbb{Q}(\aleph_1, j(\lambda), N, N[H_{j(\mathbb{R})}], H_{j(\mathbb{P})}) = \mathbb{M}(\aleph_1, j(\lambda), N, N[H_{j(\mathbb{R})}]) / H_{j(\mathbb{P})}$$

over  $N[H_{j(\mathbb{R})}][H_{j(\mathbb{P})}]$  and by Lemma 6.6 (1), that poset is  $\sigma$ -closed in  $N[H_{j(\mathbb{R})}][H_{j(\mathbb{P})}]$ . The function  $b$ , which is in  $\mathcal{N}_2$ , provides a cofinal branch for  $F^*$  in  $\mathcal{N}_2$ . It follows from the First Preservation Theorem that  $b \in N[H_{j(\mathbb{R})}][H_{j(\mathbb{P})}]$ , but we proved that  $b$  does not belong to that model, so we have a contradiction.  $\square$

This completes the proof of (3).

## 9. CONCLUSION

Cummings and Foreman [2] defined a model of the tree property for every  $\aleph_n$  ( $n \geq 2$ ), starting with an infinite sequence of supercompact cardinals  $\langle \kappa_n \rangle_{n < \omega}$ . Their forcing  $\mathbb{R}_\omega$  is basically an iteration with length  $\omega$  of our main forcing. We conjecture that  $\mathbb{R}_\omega$  produces a model in which every  $\aleph_n$  ( $n \geq 2$ ) satisfies even the super tree property. Yet, if we want to prove that stronger result, we have to deal with the following fact: every  $\kappa_n$ -tree in the Cummings-Foreman model appears in some intermedium stage, that is after forcing with  $\mathbb{R}_\omega \restriction m$  for some  $m$ ; in the case of a  $(\kappa_n, \mu)$ -tree, that is not necessarily true.

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